Partial Differential Equations

If the subject of ordinary differential equations is large, this is enormous. I am going to examine only one corner of it, and will develop only one tool to handle it: Separation of Variables. Another major tool is the method of characteristics and I'll not go beyond mentioning the word. When I develop a technique to handle the heat equation or the potential equation, don't think that it stops there. The same set of tools will work on the Schroedinger equation in quantum mechanics and on the wave equation in its many incarnations.

10.1 The Heat Equation

The flow of heat in one dimension is described by the heat conduction equation

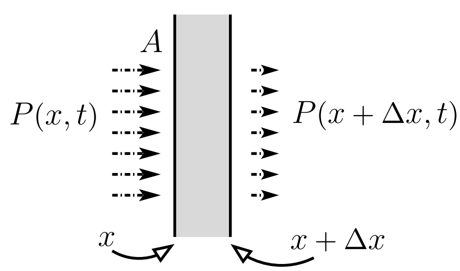
$$P = -\kappa A \frac{\partial T}{\partial x} \tag{10.1}$$

where P is the power in the form of heat energy flowing toward positive x through a wall and A is the area of the wall. κ is the wall's thermal conductivity. Put this equation into words and it says that if a thin slab of material has a temperature on one side different from that on the other, then heat energy will flow through the slab. If the temperature difference is big or the wall is thin ($\partial T/\partial x$ is big) then there's a big flow. The minus sign says that the energy flows from hot toward cold.

When more heat comes into a region than leaves it, the temperature there will rise. This is described by the specific heat, c.

$$dQ = mc\,dT, \qquad \text{or} \qquad \frac{dQ}{dt} = mc\frac{dT}{dt} \tag{10.2}$$

Again in words, the temperature rise in a chunk of material is proportional to the amount of heat added to it and inversely proportional to its mass.



For a slab of area A, thickness Δx, and mass density ρ, let the coordinates of the two sides be x and $x + \Delta x$.

$$m = \rho A \Delta x, \qquad \text{and} \qquad \frac{dQ}{dt} = P(x,t) - P(x+\Delta x, t)$$

The net power into this volume is the power in from one side minus the power out from the other. Put these three equations together.

$$\frac{dQ}{dt} = mc\frac{dT}{dt} = \rho A \Delta x\, c\frac{dT}{dt} = -\kappa A \frac{\partial T(x,t)}{\partial x} + \kappa A \frac{\partial T(x+\Delta x,t)}{\partial x}$$

If you let $\Delta x \to 0$ here, all you get is $0 = 0$, not very helpful. Instead divide by Δx first and then take the limit.

$$\frac{\partial T}{\partial t} = +\frac{\kappa A}{\rho c A}\left(\frac{\partial T(x+\Delta x,t)}{\partial x} - \frac{\partial T(x,t)}{\partial x}\right)\frac{1}{\Delta x}$$

and in the limit this is

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c\rho}\frac{\partial^2 T}{\partial x^2} \tag{10.3}$$

I was a little cavalier with the notation in that I didn't specify the argument of T on the left side. You could say that it was $(x + \Delta x/2, t)$, but in the limit everything is evaluated at (x, t) anyway. I also assumed that κ, the thermal conductivity, is independent of x. If not, then it stays inside the derivative,

$$\frac{\partial T}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) \tag{10.4}$$

In Three Dimensions

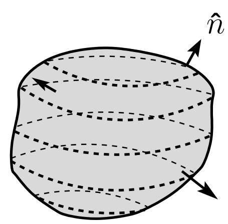
In three dimensions, this becomes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c\rho}\nabla^2 T \tag{10.5}$$

Roughly speaking, the temperature in a box can change because of heat flow in any of three directions. More precisely, the correct three dimensional equation that replaces Eq. (10.1) is

$$\vec{H} = -\kappa\nabla T \tag{10.6}$$

where $\vec{H}$ is the heat flow vector. That is the power per area in the direction of the energy transport. $\vec{H}\cdot d\vec{A} = dP$, the power going across the area $d\vec{A}$. The total heat flowing into a volume is



$$\frac{dQ}{dt} = -\oint dP = -\oint \vec{H}\cdot d\vec{A} \tag{10.7}$$

where the minus sign occurs because this is the heat flow *in.* For a small volume ΔV, you now have $m = \rho\Delta V$ and

$$mc\frac{\partial T}{\partial t} = \rho\Delta V\, c\frac{\partial T}{\partial t} = -\oint \vec{H}\cdot d\vec{A}$$

Divide by ΔV and take the limit as $\Delta V \to 0$. The right hand side is the divergence, Eq. (9.9).

$$\rho c\frac{\partial T}{\partial t} = -\lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint \vec{H}\cdot d\vec{A} = -\nabla\cdot\vec{H} = +\nabla\cdot\kappa\nabla T = +\kappa\nabla^2 T$$

The last step again assumes that the thermal conductivity, κ, is independent of position.

10.2 Separation of Variables

How do you solve these equations? I'll start with the one-dimensional case and use the method of *separation of variables.* The trick is to start by looking for a solution to the equation in the form of a product of a function of x and a function of t. $T(x, t) = f(t)g(x)$. *I do not* assume that every solution to the equation will look like this — that's just not true. What will happen is that I'll be able to express every solution as a sum of such factored forms. That this is so is a theorem that I don't plan to prove here. For that you should go to a purely mathematical text on PDEs.

If you want to find out if you have a solution, plug in:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c\rho}\frac{\partial^2 T}{\partial x^2} \qquad \text{is} \qquad \frac{df}{dt}g = \frac{\kappa}{c\rho}f\frac{d^2 g}{dx^2}$$

Denote the constant by $\kappa/c\rho = D$ and divide by the product fg.

$$\frac{1}{f}\frac{df}{dt} = D\frac{1}{g}\frac{d^2 g}{dx^2} \tag{10.8}$$

The left side of this equation is a function of t alone, no x. The right side is a function of x alone with no t, hence the name separation of variables. Because x and t can vary quite independently of each other, the only way that this can happen is if the two side are constant (the same constant).

$$\frac{1}{f}\frac{df}{dt} = \alpha \qquad \text{and} \qquad D\frac{1}{g}\frac{d^2 g}{dx^2} = \alpha \tag{10.9}$$

At this point, the constant α can be anything, even complex. For a particular specified problem there will be boundary conditions placed on the functions, and those will constrain the α's. If α is real and positive then

$$g(x) = A\sinh\sqrt{\alpha/D}\,x + B\cosh\sqrt{\alpha/D}\,x \qquad \text{and} \qquad f(t) = e^{\alpha t} \tag{10.10}$$

For negative real α, the hyperbolic functions become circular functions.

$$g(x) = A\sin\sqrt{-\alpha/D}\,x + B\cos\sqrt{-\alpha/D}\,x \qquad \text{and} \qquad f(t) = e^{\alpha t} \tag{10.11}$$

If $\alpha = 0$ then

$$g(x) = Ax + B, \qquad \text{and} \qquad f(t) = \text{ constant} \tag{10.12}$$

For imaginary α the $f(t)$ is oscillating and the $g(x)$ has both exponential and oscillatory behavior in space. This can really happen in very ordinary physical situations; see section 10.3.

This analysis provides a solution to the original equation (10.3) valid for any α. A sum of such solutions for different α's is also a solution, for example

$$T(x,t) = A_1 e^{\alpha_1 t}\sin\sqrt{-\alpha_1/D}\,x + A_2 e^{\alpha_2 t}\sin\sqrt{-\alpha_2/D}\,x$$

or any other linear combination with various α's

$$T(x,t) = \sum_{\{\alpha'\text{s}\}} f_\alpha(t) g_\alpha(x)$$

It is the combined product that forms a solution to the original partial differential equation, not the separate factors. Determining the details of the sum is a job for Fourier series.

Example

A specific problem: You have a slab of material of thickness L and at a uniform temperature T_0. Plunge it into ice water at temperature $T = 0$ and find the temperature inside at later times. The boundary condition here is that the surface temperature is zero, $T(0,t) = T(L,t) = 0$. This constrains the separated solutions, requiring that $g(0) = g(L) = 0$. For this to happen you can't use the hyperbolic

functions of x that occur when $\alpha > 0$, you will need the circular functions of x, sines and cosines, implying that $\alpha < 0$. That is also compatible with your expectation that the temperature should approach zero eventually, and that needs a negative exponential in time, Eq. (10.11).

$$g(x) = A\sin kx + B\cos kx, \qquad \text{with} \qquad k^2 = -\alpha/D \qquad \text{and} \qquad f(t) = e^{-Dk^2 t}$$

$g(0) = 0$ implies $B = 0$. $g(L) = 0$ implies $\sin kL = 0$.
The sine vanishes for the values $n\pi$ where n is any integer, positive, negative, or zero. This implies $kL = n\pi$, or $k = n\pi/L$. The corresponding values of α are $\alpha_n = -Dn^2\pi^2/L^2$, and the separated solution is

$$\sin\big(n\pi x/L\big)\, e^{-n^2\pi^2 Dt/L^2} \tag{10.13}$$

If $n = 0$ this whole thing vanishes, so it's not much of a solution. (Not so fast there! See problem 10.2.) Notice that the sine is an odd function so when $n < 0$ this expression just reproduces the positive n solution except for an overall factor of (-1), and that factor was arbitrary anyway. The negative n solutions are redundant, so ignore them.

The general solution is a sum of separated solutions, see problem 10.3.

$$T(x,t) = \sum_1^\infty a_n \sin\frac{n\pi x}{L}\, e^{-n^2\pi^2 Dt/L^2} \tag{10.14}$$

The problem now is to determine the coefficients a_n. *This* is why Fourier series were invented. (Yes, literally, the problem of heat conduction is where Fourier series started.) At time $t = 0$ you know the temperature distribution is $T = T_0$, a constant on $0 < x < L$. This general sum must equal T_0 at time $t = 0$.

$$T(x,0) = \sum_1^\infty a_n \sin\frac{n\pi x}{L} \qquad (0 < x < L)$$

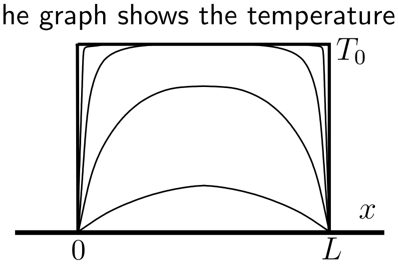
Multiply by $\sin\big(m\pi x/L\big)$ and integrate over the domain to isolate the single term, $n = m$.

$$\begin{aligned} \int_0^L dx\, T_0 \sin\frac{m\pi x}{L} &= a_m \int_0^L dx\, \sin^2\frac{m\pi x}{L} \\ T_0[1 - \cos m\pi]\frac{L}{m\pi} &= a_m \frac{L}{2} \end{aligned}$$

This expression for a_m vanishes for even m, and when you assemble the whole series for the temperature you have

$$T(x,t) = \frac{4}{\pi} T_0 \sum_{m\ \text{odd}} \frac{1}{m} \sin\frac{m\pi x}{L} e^{-m^2\pi^2 Dt/L^2} \tag{10.15}$$

For small time, this converges, but very slowly. For large time, the convergence is very fast, often needing only one or two terms. As the time approaches infinity, the interior temperature approaches the surface temperature of zero. The graph shows the temperature profile at a sequence of times.



The curves show the temperature dropping very quickly for points near the surface ($x = 0$ or L). It drops more gradually near the center but eventually goes to zero.

You can see that the boundary conditions on the temperature led to these specific boundary conditions on the sines and cosines. This is exactly what happened in the general development of Fourier series when the fundamental relationship, Eq. (5.15), required certain boundary conditions in order to get the orthogonality of the solutions of the harmonic oscillator differential equation. That the function vanishes at the boundaries was one of the possible ways to insure orthogonality.

10.3 Oscillating Temperatures

Take a very thick slab of material and assume that the temperature on one side of it is oscillating. Let the material occupy the space $0 < x < \infty$ and at the coordinate $x = 0$ the temperature is varying in time as $T_1 \cos \omega t$. Is there any real situation in which this happens? Yes, the surface temperature of the Earth varies periodically from summer to winter (even in Florida). What happens to the temperature underground?

The differential equation for the temperature is still Eq. (10.3), and assume that the temperature inside the material approaches $T = 0$ far away from the surface. Separation of variables is the same as before, Eq. (10.9), but this time you know the time dependence at the surface. It's typical in cases involving oscillations that it is easier to work with complex exponentials than it is to work with sines and cosines. For that reason, specify that the surface temperature is $T_1 e^{-i\omega t}$ instead of a cosine, understanding that at the end of the problem you must take the real part of the result and throw away the imaginary part. The imaginary part corresponds to solving the problem for a surface temperature of $\sin \omega t$ instead of cosine. It's easier to solve the two problems together then either one separately. (The minus sign in the exponent of $e^{-i\omega t}$ is arbitrary; you could use a plus instead.)

The equation (10.9) says that the time dependence that I expect is

$$\frac{1}{f}\frac{df}{dt} = \alpha = \frac{1}{e^{-i\omega t}}\big(-i\omega e^{-i\omega t}\big) = -i\omega$$

The equation for the x-dependence is then

$$D\frac{d^2 g}{dx^2} = \alpha g = -i\omega g$$

This is again a simple exponential solution, say $e^{\beta x}$. Substitute and you have

$$D\beta^2 e^{\beta x} = -i\omega e^{\beta x}, \qquad \text{implying} \qquad \beta = \pm\sqrt{-i\omega/D} \tag{10.16}$$

Evaluate this as

$$\sqrt{-i} = \left(e^{-i\pi/2}\right)^{1/2} = e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$$

Let $\beta_0 = \sqrt{\omega/2D}$, then the solution for the x-dependence is

$$g(x) = Ae^{(1-i)\beta_0 x} + Be^{-(1-i)\beta_0 x} \tag{10.17}$$

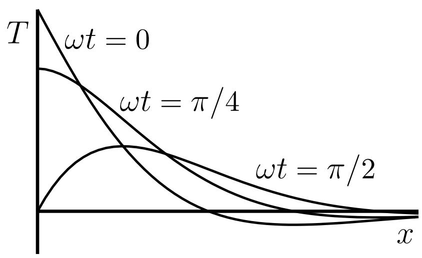
Look at the behavior of these two terms. The first has a factor that goes as e^{+x} and the second goes as e^{-x}. The temperature at large distances is supposed to approach zero, so that says that $A = 0$. The solutions for the temperature is now

$$Be^{-i\omega t}e^{-(1-i)\beta_0 x} \tag{10.18}$$

The further condition is that at $x = 0$ the temperature is $T_1 e^{-i\omega t}$, so that tells you that $B = T_1$.

$$T(x,t) = T_1 e^{-i\omega t} e^{-(1-i)\beta_0 x} = T_1 e^{-\beta_0 x} e^{i(-\omega t + \beta_0 x)} \tag{10.19}$$

When you remember that I'm solving for the real part of this solution, the final result is



$$T_1 e^{-\beta_0 x} \cos(\beta_0 x - \omega t) \tag{10.20}$$

This has the appearance of a temperature wave moving into the material, albeit a very strongly damped one. In a half wavelength of this wave, $\beta_0 x = \pi$, and at that point the amplitude coming from the exponential factor out in front is down by a factor of $e^{-\pi} = 0.04$. That's barely noticeable. This is why wine cellars are cellars. Also, you can see that at a distance where $\beta_0 x > \pi/2$ the temperature change is reversed from the value at the surface. Some distance underground, summer and winter are reversed. This same sort of equation comes up with the study of eddy currents in electromagnetism, so the same sort of results obtain.

10.4 Spatial Temperature Distributions

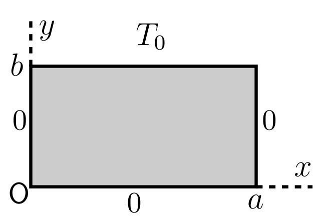
The governing equation is Eq. (10.5). For an example of a problem that falls under this heading, take a cube that is heated on one side and cooled on the other five sides. What is the temperature distribution within the cube? How does it vary in time?

I'll take a simpler version of this problem to start with. First, I'll work in two dimensions instead of three; make it a very long rectangular shaped rod, extending in the z-direction. Second, I'll look for the equilibrium solution, for which the time derivative is zero. These restrictions reduce the equation (10.5) to

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{10.21}$$

I'll specify the temperature $T(x,y)$ on the surface of the rod to be zero on three faces and T_0 on the fourth. Place the coordinates so that the length of the rod is along the z-axis and the origin is in one corner of the rectangle.

$$\begin{aligned} T(0,y) &= 0 \quad (0 < y < b), \qquad & T(x,0) &= 0 \quad (0 < x < a) \\ T(a,y) &= 0 \quad (0 < y < b), \qquad & T(x,b) &= T_0 \quad (0 < x < a) \end{aligned} \tag{10.22}$$



Look at this problem from several different angles, tear it apart, look at a lot of special cases, and see what can go wrong. In the process you'll see different techniques and especially a lot of applications of Fourier series. This single problem will illustrate many of the methods used to understand boundary value problems.

Use the same method used before for heat flow in one dimension: separation of variables. Assume a solution to be the product of a function of x and a function of y, then plug into the equation.

$$T(x,y) = f(x)g(y), \qquad \text{then} \qquad \nabla^2 T = \frac{d^2 f(x)}{dx^2} g(y) + f(x)\frac{d^2 g(y)}{dy^2} = 0$$

Just as in Eq. (10.8), when you divide by fg the resulting equation is separated into a term involving x only and one involving y only.

$$\frac{1}{f}\frac{d^2 f(x)}{dx^2} + \frac{1}{g}\frac{d^2 g(y)}{dy^2} = 0$$

Because x and y can be varied independently, these must be constants adding to zero.

$$\frac{1}{f}\frac{d^2 f(x)}{dx^2} = \alpha, \qquad \text{and} \qquad \frac{1}{g}\frac{d^2 g(y)}{dy^2} = -\alpha \tag{10.23}$$

As before, the separation constant can be any real or complex number until you start applying boundary conditions. You recognize that the solutions to these equations can be sines or cosines or exponentials or hyperbolic functions or linear functions, depending on what α is.

The boundary conditions state that the surface temperature is held at zero on the surfaces $x = 0$ and $x = a$. This suggests looking for solutions that vanish there, and that in turn says you should work with sines of x. In the other direction the surface temperature vanishes on only one side so you don't need sines in that case. The $\alpha = 0$ case gives linear functions is x and in y, and the fact that the temperature vanishes on $x = 0$ and $x = a$ kills these terms. (It does doesn't it?) Pick α to be a negative real number: call it $\alpha = -k^2$.

$$\frac{d^2 f(x)}{dx^2} = -k^2 f \qquad \Longrightarrow \qquad f(x) = A\sin kx + B\cos kx$$

The accompanying equation for g is now

$$\frac{d^2 g(y)}{dy^2} = +k^2 g \qquad \Longrightarrow \qquad g(y) = C\sinh ky + D\cosh ky$$

(Or exponentials if you prefer.) The combined, separated solution to $\nabla^2 T = 0$ is

$$(A\sin kx + B\cos kx)(C\sinh ky + D\cosh ky) \tag{10.24}$$

The general solution will be a sum of these, summed over various values of k. This is where you have to apply the boundary conditions to determine the allowed k's.

$$\text{left:}\quad T(0,y) = 0 = B(C\sinh ky + D\cosh ky), \qquad \text{so} \qquad B = 0$$

(This holds for all y in $0 < y < b$, so the second factor can't vanish unless both C and D vanish. If that is the case then *everything* vanishes.)

$$\text{right:}\quad T(a,y) = 0 = A\sin ka(C\sinh ky + D\cosh ky), \qquad \text{so} \qquad \sin ka = 0$$

(The factor with y can't vanish or everything vanishes. If $A = 0$ then everything vanishes. All that's left is $\sin ka = 0$.)

$$\text{bottom:}\quad T(x,0) = 0 = A\sin kx\, D, \qquad \text{so} \qquad D = 0$$

(If $A = 0$ everything is zero, so it has to be D.)

You can now write a general solution that satisfies three of the four boundary conditions. Combine the coefficients A and C into one, and since it will be different for different values of k, call it γ_n.

$$T(x,y) = \sum_{n=1}^{\infty} \gamma_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi y}{a} \tag{10.25}$$

The $n\pi/a$ appears because $\sin ka = 0$, and the limits on n omit the negative n because they are redundant.

Now to find all the unknown constants γ_n, and as before that's where Fourier techniques come in. The fourth side, at $y = b$, has temperature T_0 and that implies

$$\sum_{n=1}^{\infty} \gamma_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi b}{a} = T_0$$

On this interval $0 < x < a$ these sine functions are orthogonal, so you take the scalar product of both side with the sine.

$$\int_0^a dx\,\sin\frac{m\pi x}{a}\sum_{n=1}^{\infty} \gamma_n \sin\frac{n\pi x}{a}\,\sinh\frac{n\pi b}{a} = \int_0^a dx\,\sin\frac{m\pi x}{a}\,T_0$$

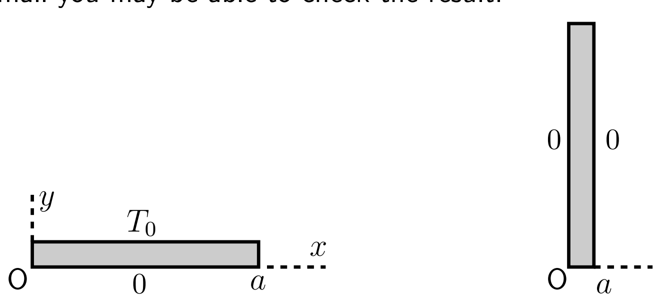
$$\frac{a}{2}\gamma_m \sinh\frac{m\pi b}{a} = T_0\frac{a}{m\pi}\big[1-(-1)^m\big]$$

Just the odd m terms are present, $m = 2\ell+1$, so the result is

$$T(x,y) = \frac{4}{\pi}T_0\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\frac{\sinh\big((2\ell+1)\pi y/a\big)}{\sinh\big((2\ell+1)\pi b/a\big)}\sin\frac{(2\ell+1)\pi x}{a} \tag{10.26}$$

You're not done.

Does this make sense? The dimensions are clearly correct, but after that it takes some work. There's really just one parameter that you have to play around with, and that's the ratio b/a. If it's either very big or very small you may be able to check the result.



$a \gg b$ $\qquad$ $b \gg a$

If $a \gg b$, it looks almost like a one-dimensional problem. It is a thin slab with temperature T_0 on one side and zero on the other. There's little variation along the x-direction, so the equilibrium equation is

$$\nabla^2 T = 0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \approx \frac{\partial^2 T}{\partial y^2}$$

This simply says that the second derivative with respect to y vanishes, so the answer is the straight line $T = A + By$, and with the condition that you know the temperature at $y = 0$ and at $y = b$ you easily find

$$T(x, y) \approx T_0 y/b$$

Does the exact answer look like this? It doesn't seem to, but look closer. If $b \ll a$ then because $0 < y < b$ you also have $y \ll a$. The hyperbolic function factors in Eq. (10.26) will have very small arguments, proportional to b/a. Recall the power series expansion of the hyperbolic sine: $\sinh x = x + \cdots$. These factors become approximately

$$\frac{\sinh\big((2\ell+1)\pi y/a\big)}{\sinh\big((2\ell+1)\pi b/a\big)} \approx \frac{(2\ell+1)\pi y/a}{(2\ell+1)\pi b/a} = \frac{y}{b}$$

The temperature solution is then

$$T(x, y) \approx \frac{4}{\pi} T_0 \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \frac{y}{b} \sin\frac{(2\ell+1)\pi x}{a} = T_0 \frac{y}{b}$$

Where did that last equation come from? The coefficient of y/b is just the Fourier series of the constant T_0 in terms of sines on $0 < x < a$.

What about the opposite extreme, for which $b \gg a$? This is the second picture just above. Instead of being short and wide it is tall and narrow. For this case, look again at the arguments of the hyperbolic sines. Now $\pi b/a$ is large and you can approximate the hyperbolic functions by going back to their definition.

$$\sinh x = \frac{e^x + e^{-x}}{2} \approx \frac{1}{2} e^x, \quad \text{for } x \gg 1$$

The denominators in all the terms of Eq. (10.26) are large, $\approx e^{\pi b/a}$ (or larger still because of the $(2\ell + 1)$). This will make all the terms in the series extremely small *unless* the numerators are correspondingly large. This means that the temperature stays near zero unless y is large. That makes sense. It's only for y near the top end that you are near to the wall with temperature T_0.

You now have the case for which $b \gg a$ and $y \gg a$. This means that I can use the approximate form of the hyperbolic function for large arguments.

$$\frac{\sinh\big((2\ell+1)\pi y/a\big)}{\sinh\big((2\ell+1)\pi b/a\big)} \approx \frac{e^{(2\ell+1)\pi y/a}}{e^{(2\ell+1)\pi b/a}} = e^{(2\ell+1)\pi(y-b)/a}$$

The temperature distribution is now approximately

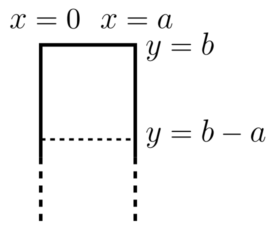
$$T(x, y) \approx \frac{4}{\pi} T_0 \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} e^{-(2\ell+1)\pi(b-y)/a} \sin\frac{(2\ell+1)\pi x}{a} \tag{10.27}$$

As compared to the previous approximation where $a \gg b$, you can't as easily tell whether this is plausible or not. You can however learn from it. See also problem 10.30.

At the very top, where $y = b$ this reduces to the constant T_0 that you're supposed to have at that position. Recall the Fourier series for a constant on $0 < x < a$.

Move down from $y = b$ by the distance a, so that $b - y = a$. That's a distance from the top equal to the width of the rectangle. It's still rather close to the end, but look at the series for that

position.



$$T(x,b-a) \approx \frac{4}{\pi} T_0 \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} e^{-(2\ell+1)\pi} \sin\frac{(2\ell+1)\pi x}{a}$$

For $\ell = 0$, the exponential factor is $e^{-\pi} = 0.043$, and for $\ell = 1$ this factor is $e^{-3\pi} = 0.00008$. This means that measured from the T_0 end, within the very short distance equal to the width, the temperature has dropped 95% of the way down to its limiting value of zero. The temperature in the rod is quite uniform until you are very close to the heated end.

The Heat Flow into the Box

All the preceding analysis and discussion was intended to make this problem and its solution sound oh-so-plausible. There's more, and it isn't pretty.

The temperature on one of the four sides was given as different from the temperatures on the other three sides. What will the heat flow into the region be? That is, what power must you supply to maintain the temperature T_0 on the single wall?

At the beginning of this chapter, Eq. (10.1), you have the equation for the power through an area A, but that equation assumed that the temperature gradient $\partial T/\partial x$ is the same all over the area A. If it isn't, you simply turn it into a density.

$$\Delta P = -\kappa \Delta A \frac{\partial T}{\partial x}, \qquad \text{and then} \qquad \frac{\Delta P}{\Delta A} \to \frac{dP}{dA} = -\kappa \frac{\partial T}{\partial x} \tag{10.28}$$

Equivalently, just use the vector form from Eq. (10.6), $\vec{H} = -\kappa \nabla T$. In Eq. (10.22) the temperature is T_0 along $y = b$, and the power density (energy / (time · area)) flowing in the $+y$ direction is $-\kappa \partial T/\partial y$, so the power density flowing *into* this area has the reversed sign,

$$+\kappa\, \partial T/\partial y \tag{10.29}$$

The total power flow is the integral of this over the area of the top face.

Let L be the length of this long rectangular rod, its extent in the z-direction. The element of area along the surface at $y = b$ is then $dA = L dx$, and the power flow into this face is

$$\int_0^a L\, dx\, \kappa \left.\frac{\partial T}{\partial y}\right|_{y=b}$$

The temperature function is the solution Eq. (10.26), so differentiate that equation with respect to y.

$$\begin{aligned}
\int_0^a L\, dx\, \kappa \frac{4}{\pi} T_0 \sum_{\ell=0}^{\infty} \frac{[(2\ell+1)\pi/a]}{2\ell+1} \frac{\cosh\big((2\ell+1)\pi y/a\big)}{\sinh\big((2\ell+1)\pi b/a\big)} \sin\frac{(2\ell+1)\pi x}{a} \quad \text{at} \quad y=b \\
= \frac{4L\kappa T_0}{a} \int_0^a dx \sum_{\ell=0}^{\infty} \sin\frac{(2\ell+1)\pi x}{a}
\end{aligned}$$

and this sum does not converge. I'm going to push ahead anyway, temporarily pretending that I didn't notice this minor difficulty with the series. Just go ahead and integrate the series term by term and hope for the best.

$$
\begin{aligned}
&= \frac{4L\kappa T_0}{a}\sum_{\ell=0}^{\infty}\frac{a}{\pi(2\ell+1)}\big[-\cos\big((2\ell+1)\pi\big)+1\big] \\
&= \frac{4L\kappa T_0}{\pi}\sum_{\ell=0}^{\infty}\frac{2}{2\ell+1}=\infty
\end{aligned}
$$

This infinite series for the total power entering the top face is infinite. The series doesn't converge (use the integral test).

This innocuous-seeming problem is suddenly pathological because it would take an infinite power source to maintain this temperature difference. Why should that be? Look at the corners. You're trying to maintain a non-zero temperature difference $(T_0 - 0)$ between two walls that are touching. This can't happen, and the equations are telling you so! It means that the boundary conditions specified in Eq. (10.22) are impossible to maintain. The temperature on the boundary at $y = b$ can't be constant all the way to the edge. It must drop off to zero as it approaches $x = 0$ and $x = a$. This makes the problem more difficult, but then reality is typically more complicated than our simple, idealized models.

Does this make the solution Eq. (10.26) valueless? No, it simply means that you can't push it too hard. This solution will be good until you get near the corners, where you can't possibly maintain the constant-temperature boundary condition. In other regions it will be a good approximation to the physical problem.

10.5 Specified Heat Flow

In the previous examples, I specified the temperature on the boundaries and from that I determined the temperature inside. In the particular example, the solution was not physically plausible all the way to the edge, though the mathematics were (I hope) enlightening. Instead, I'll reverse the process and try to specify the size of the heat flow, computing the resulting temperature from that. This time perhaps the results will be a better reflection of reality.

Equation (10.29) tells you the power density at the surface, and I'll examine the case for which this is a constant. Call it F_0. (There's not a conventional symbol, so this will do.) The plus sign occurs because the flow is into the box.

$$+\kappa\frac{\partial T}{\partial y}(x,b) = F_0$$

The other three walls have the same zero temperature conditions as Eq. (10.22). Which forms of the separated solutions must I use now? The same ones as before or different ones?

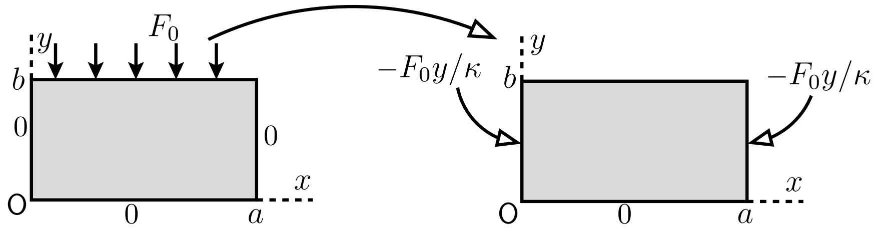
Look again at the $\alpha = 0$ solutions to Eqs. (10.23). That solution is

$$(A+Bx)(C+Dy)$$

In order to handle the fact that the temperature is zero at $y = 0$ and that the derivative with respect to y is given at $y = b$,

$$
\begin{gathered}
(A+Bx)(C)=0 \qquad \text{and} \qquad (A+Bx)(D)=F_0/\kappa, \\
\text{implying} \qquad C=0=B, \qquad \text{then} \qquad AD=F_0/\kappa \quad\Longrightarrow\quad \frac{F_0}{\kappa}y
\end{gathered}
\tag{10.30}
$$

This matches the boundary conditions at both $y = 0$ and $y = b$. All that's left is to make everything work at the other two faces.



If I can find a solution that equals $-F_0y/\kappa$ on the left and right faces then it will cancel the $+F_0y/\kappa$ that Eq. (10.30) provides. *But* I can't disturb the top and bottom boundary conditions. The way to do that is to find functions that equal zero at $y = 0$ and whose derivative equals zero at $y = b$. This is a familiar sort of condition that showed up several times in chapter five on Fourier series. It is equivalent to saying that the top surface is insulated so that heat can't flow through it. You then use superposition to combine the solution with uniform heat flow and the solution with an insulated boundary.

Instead of Eq. (10.24), use the opposite sign for α, so the solutions are of the form

$$(A\sin ky + B\cos ky)(C\sinh kx + D\cosh kx)$$

I require that this equals zero at $y = 0$, so that says

$$(0 + B)(C\sinh kx + D\cosh kx) = 0$$

so $B = 0$. Now require that the derivative equals zero at $y = b$, so

$$Ak\cos kb = 0, \qquad \text{or} \qquad kb = (n + \tfrac{1}{2})\pi \qquad \text{for} \qquad n = 0,\,1,\,2\ldots$$

The value of the temperature is the same on the left that it is on the right, so

$$C\sinh k0 + D\cosh k0 = C\sinh ka + D\cosh ka \implies C = D(1 - \cosh ka)/\sinh ka \tag{10.31}$$

This is starting to get messy, so it's time to look around and see if I've missed anything that could simplify the calculation. There's no guarantee that there is any simpler way, but it is always worth looking. The fact that the system is the same on the left as on the right means that the temperature will be symmetric about the central axis of the box, about $x = a/2$. That it is even about this point implies that the hyperbolic functions of x should be even about $x = a/2$. You can do this simply by using a $\cosh$ about that point.

$$A\sin ky\big(D\cosh k(x - \tfrac{a}{2})\big)$$

Put these together and you have a sum

$$\sum_{n=0}^{\infty} a_n \sin\left(\frac{(n+\frac{1}{2})\pi y}{b}\right)\cosh\left(\frac{(n+\frac{1}{2})\pi(x-\frac{a}{2})}{b}\right) \tag{10.32}$$

Each of these terms satisfies Laplace's equation, satisfies the boundary conditions at $y = 0$ and $y = b$, and is even about the centerline $x = a/2$. It is now a problem in Fourier series to match the conditions at $x = 0$. They're then automatically satisfied at $x = a$.

$$\sum_{n=0}^{\infty} a_n \sin\left(\frac{(n+\frac{1}{2})\pi y}{b}\right)\cosh\left(\frac{(n+\frac{1}{2})\pi a}{2b}\right) = -F_0\frac{y}{\kappa} \tag{10.33}$$

The sines are orthogonal by the theorem Eq. (5.15), so you can pick out the component a_n by the orthogonality of these basis functions.

$$u_n = \sin\left(\frac{(n+\frac{1}{2})\pi y}{b}\right), \qquad \text{then} \qquad \langle u_m, \text{left side}\rangle = \langle u_m, \text{right side}\rangle$$

$$\text{or,} \qquad a_m \langle u_m, u_m\rangle \cosh\left(\frac{(m+\frac{1}{2})\pi a}{2b}\right) = -\frac{F_0}{\kappa}\langle u_m, y\rangle$$

Write this out; do the integrals, add the linear term, and you have

$$T(x,y) = F_0\frac{y}{\kappa} - \frac{8F_0 b}{\kappa\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\times \\ \sin\left(\frac{(n+\frac{1}{2})\pi y}{b}\right)\cosh\left(\frac{(n+\frac{1}{2})\pi(x-\frac{a}{2})}{b}\right)\operatorname{sech}\left(\frac{(n+\frac{1}{2})\pi a}{2b}\right) \tag{10.34}$$

Now analyze this to see if it makes sense. I'll look at the same cases as the last time: $b \ll a$ and $a \ll b$. The simpler case, where the box is short and wide, has $b \ll a$. This makes the arguments of the cosh and sech large, with an a/b in them. For large argument you can approximate the cosh by

$$\cosh x \approx e^x/2, \qquad x \gg 1$$

Now examine a typical term in the sum (10.34), and I have to be a little more specific and choose x on the left or right of $a/2$. The reason for that is the preceding equation requires x large and positive. I'll take x on the right, as it makes no difference. The hyperbolic functions in (10.34) are approximately

$$\frac{\exp\left((n+\frac{1}{2})\pi(x-\frac{a}{2})/b\right)}{\exp\left((n+\frac{1}{2})\pi a/2b\right)} = e^{\left((2n+1)\pi(x-a)/2b\right)}$$

As long as x is not near the end, that is, not near $x = a$, the quantity in the exponential is large and negative for all n. The exponential in turn makes this extremely small so that the entire sum becomes negligible. The temperature distribution is then the single term

$$T(x,y) \approx F_0\frac{y}{\kappa}$$

It's essentially a one dimensional problem, with the heat flow only along the $-y$ direction.

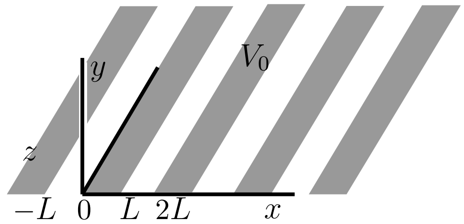
In the reverse case for which the box is tall and thin, $a \ll b$, the arguments of the hyperbolic functions are small. This invites a power series expansion, but that approach doesn't work. The analysis of this case is quite tricky, and I finally concluded that it's not worth the trouble to write it up. It leads to a rather complicated integral.

10.6 Electrostatics

The equation for the electrostatic potential in a vacuum is exactly the same as Eq. (10.21) for the temperature in static equilibrium, $\nabla^2 V = 0$, with the electric field $\vec{E} = -\nabla V$. The same equation applies to the gravitational potential, Eq. (9.42).

Perhaps you've looked into a microwave oven. You can see inside it, but the microwaves aren't supposed to get out. How can this be? Light is just another form of electromagnetic radiation, so why

does one EM wave get through while the other one doesn't? I won't solve the whole electromagnetic radiation problem here, but I'll look at the static analog to get some general idea of what's happening.



Arrange a set of conducting strips in the x-y plane and with insulation between them so that they don't quite touch each other. Now apply voltage V_0 on every other one so that the potentials are alternately zero and V_0. This sets the potential in the $z=0$ plane to be independent of y and

$$z=0: \quad V(x,y)=\begin{cases} V_0 & (0<x<L) \\ 0 & (L<x<2L) \end{cases} \qquad V(x+2L,y)=V(x,y), \text{ all } x,\, y \tag{10.35}$$

What is then the potential above the plane, $z>0$? Above the plane V satisfies Laplace's equation,

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{10.36}$$

The potential is independent of y in the plane, so it will be independent of y everywhere. Separate variables in the remaining coordinates.

$$V(x,z)=f(x)g(z)\Longrightarrow \frac{d^2 f}{dx^2}g+f\frac{d^2 g}{dz^2}=0\Longrightarrow \frac{1}{f}\frac{d^2 f}{dx^2}+\frac{1}{g}\frac{d^2 g}{dz^2}=0$$

This is separated as a function of x plus a function of y, so the terms are constants.

$$\frac{1}{f}\frac{d^2 f}{dx^2}=-\alpha^2, \qquad \frac{1}{g}\frac{d^2 g}{dz^2}=+\alpha^2 \tag{10.37}$$

I've chosen the separation constant in this form because the boundary condition is periodic in x, and that implies that I'll want oscillating functions there, not exponentials.

$$f(x)=e^{i\alpha x} \qquad \text{and} \qquad f(x+2L)=f(x)$$
$$\Longrightarrow e^{2Li\alpha}=1, \qquad \text{or} \qquad 2L\alpha=2n\pi,\ n=0,\,\pm1,\,\pm2,\ldots$$

The separated solutions are then

$$f(x)g(z)=e^{n\pi ix/L}\big(Ae^{n\pi z/L}+Be^{-n\pi z/L}\big) \tag{10.38}$$

The solution for $z>0$ is therefore the sum

$$V(x,z)=\sum_{n=-\infty}^{\infty}e^{n\pi ix/L}\big(A_n e^{n\pi z/L}+B_n e^{-n\pi z/L}\big) \tag{10.39}$$

The coefficients A_n and B_n are to be determined by Fourier techniques. First however, look at the z-behavior. As you move away from the plane toward positive z, the potential should not increase

without bound. Terms such as $e^{\pi z/L}$ however increase with z. This means that the coefficients of the terms that increase exponentially in z cannot be there.

$$A_n = 0 \text{ for } n > 0, \qquad \text{and} \qquad B_n = 0 \text{ for } n < 0$$

$$V(x,z) = A_0 + B_0 + \sum_{n=1}^{\infty} e^{n\pi i x/L} B_n e^{-n\pi z/L} + \sum_{n=-\infty}^{-1} e^{n\pi i x/L} A_n e^{n\pi z/L} \tag{10.40}$$

The combined constant $A_0 + B_0$ is really one constant; you can call it C_0 if you like. Now use the usual Fourier techniques given that you know the potential at $z = 0$.

$$V(x,0) = C_0 + \sum_{n=1}^{\infty} B_n e^{n\pi i x/L} + \sum_{n=-\infty}^{-1} A_n e^{n\pi i x/L}$$

The scalar product of $e^{m\pi i x/L}$ with this equation is

$$\big\langle e^{m\pi i x/L}, V(x,0)\big\rangle = \begin{cases} 2LC_0 & (m=0) \\ 2LB_m & (m>0) \\ 2LA_m & (m<0) \end{cases} \tag{10.41}$$

Now evaluate the integral on the left side. First, $m \neq 0$:

$$\begin{aligned} \big\langle e^{m\pi i x/L}, V(x,0)\big\rangle &= \int_{-L}^{L} dx\, e^{-m\pi i x/L} \begin{cases} 0 & (-L<x<0) \\ V_0 & (0<x<L) \end{cases} \\ &= V_0 \int_0^L dx\, e^{-m\pi i x/L} = V_0 \frac{L}{-m\pi i}\, e^{-m\pi i x/L}\Big|_0^L \\ &= V_0 \frac{L}{-m\pi i}\big[(-1)^m - 1\big] \end{aligned}$$

Then evaluate it separately for $m = 0$, and you have $\big\langle 1, V(x,0)\big\rangle = V_0 L$.

Now assemble the result. Before plunging in, look at what will happen.
The $m = 0$ term sits by itself.
For the other terms, only odd m have non-zero values.

$$\begin{aligned} V(x,z) = \frac{1}{2}V_0 &+ V_0 \sum_{m=1}^{\infty} \frac{1}{-2m\pi i}\big[(-1)^m - 1\big] e^{m\pi i x/L} e^{-m\pi z/L} \\ &+ V_0 \sum_{m=-\infty}^{-1} \frac{1}{-2m\pi i}\big[(-1)^m - 1\big] e^{m\pi i x/L} e^{+m\pi z/L} \end{aligned} \tag{10.42}$$

To put this into a real form that is easier to interpret, change variables, letting $m = -n$ in the second sum and $m = n$ in the first, finally changing the sum so that it is over just the odd terms.

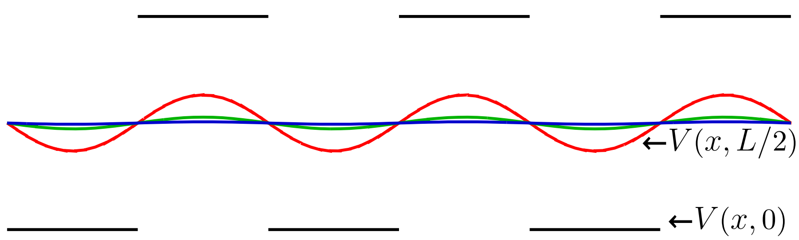
$$\begin{aligned} V(x,z) &= \frac{1}{2}V_0 + V_0 \sum_{n=1}^{\infty} \frac{1}{-2n\pi i}\big[(-1)^n - 1\big] e^{n\pi i x/L} e^{-n\pi z/L} \\ &\qquad + V_0 \sum_{1}^{\infty} \frac{1}{+2n\pi i}\big[(-1)^n - 1\big] e^{-n\pi i x/L} e^{-n\pi z/L} \\ &= \frac{1}{2}V_0 + V_0 \sum_{n=1}^{\infty} \big[(-1)^n - 1\big] \frac{1}{-n\pi} \sin(n\pi x/L) e^{-n\pi z/L} \\ &= \frac{1}{2}V_0 + \frac{2}{\pi}V_0 \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \sin\big((2\ell+1)\pi x/L\big) e^{-(2\ell+1)\pi z/L} \end{aligned} \tag{10.43}$$

Having done all the work to get to the answer, what can I learn from it?
What does it look like?
Are there any properties of the solution that are unexpected?
Should I have anticipated the form of the result?
Is there an easier way to get to the result?

To see what it looks like, examine some values of z, the distance above the surface. If $z = L$, the coefficient for successive terms is

$$\ell = 0: \ \frac{2}{\pi}e^{-\pi} = 0.028 \qquad\qquad \ell = 1: \ \frac{2}{3\pi}e^{-3\pi} = 1.7\times 10^{-5} \tag{10.44}$$

The constant term is the average potential, and the $\ell = 0$ term adds only a modest ripple, about 5% of the constant average value. If you move up to $z = 2L$ the first factor is 0.0012 and that's a little more than 0.2% ripple. The sharp jumps from $+V_0$ to zero and back disappear rapidly. That the oscillations vanish so quickly with distance is perhaps not what you would guess until you have analyzed such a problem.



The graph shows the potential function at the surface, $z = 0$, as it oscillates between V_0 and zero. It then shows successive graphs of Eq. (10.43) at $z = L/2$, then at $z = L$, then at $z = 1.5L$. The ripple is barely visible at the that last distance. The radiation through the screen of a microwave oven is filtered in much the same way because the wavelength of the radiation is large compared to the size of the holes in the screen.

When you write the form of the series for the potential, Eq. (10.40), you can see this coming if you look for it. The oscillating terms in x are accompanied by exponential terms in z, and the rapid damping with distance is already apparent: $e^{-n\pi z/L}$. You don't have to solve for a single coefficient to see that the oscillations vanish very rapidly with distance.

The original potential on the surface was neither even nor odd, but except for the constant average value, it *is* an odd function.

$$z = 0: \quad V(x,y) = \frac{1}{2}V_0 + \begin{cases} +V_0/2 & (0 < x < L) \\ -V_0/2 & (L < x < 2L) \end{cases} \qquad V(x + 2L, y) = V(x, y) \tag{10.45}$$

Solve the potential problem for the constant $V_0/2$ and you have a constant. Solve it for the remaining odd function on the boundary and you should expect an odd function for $V(x, z)$. If you make these observations *before* solving the problem you can save yourself some algebra, as it will lead you to the form of the solution faster.

The potential is periodic on the x-y plane, so periodic boundary conditions are the appropriate ones. You can express these in more than one way, taking as a basis for the expansion either complex exponentials or sines and cosines.

$$\begin{aligned} & e^{n\pi i x/L}, \ n = 0,\, \pm 1, \ldots \\ \text{or the combination} \quad & \cos(n\pi x/L), \ n = 0,\, 1, \ldots \qquad \sin(n\pi x/L), \ n = 1,\, 2, \ldots \end{aligned} \tag{10.46}$$

For a random problem with no special symmetry the exponential choice typically leads to easier integrals. In this case the boundary condition has some symmetry that you can take advantage of: it's almost

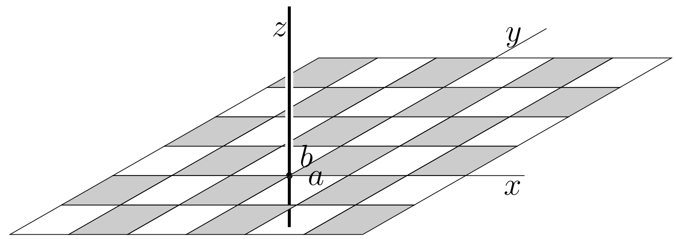
odd. The constant term in Eq. (10.30) is the $n = 0$ element of the cosine set, and that's necessarily orthogonal to all the sines. For the rest, you do the expansion

$$\begin{cases} +V_0/2 & (0 < x < L) \\ -V_0/2 & (L < x < 2L) \end{cases} = \sum_1^\infty a_n \sin(n\pi x/L)$$

The odd term in the boundary condition (10.45) is necessarily a sum of sines, with no cosines. The cosines are orthogonal to an odd function. See problem 10.11.

More Electrostatic Examples

Specify the electric potential in the x-y plane to be an array, periodic in both the x and the y-directions. $V(x, y, z = 0)$ is V_0 on the rectangle $(0 < x < a,\ 0 < y < b)$ as well as in the darkened boxes in the picture; it is zero in the white boxes. What is the potential for $z > 0$?



The equation is still Eq. (10.36), but now you have to do the separation of variables along all three coordinates, $V(x, y, z) = f(x)g(y)h(z)$. Substitute into the Laplace equation and divide by fgh.

$$\frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} + \frac{1}{h}\frac{d^2h}{dz^2} = 0$$

These terms are functions of the single variables x, y, and z respectively, so the only way this can work is if they are separately constant.

$$\frac{1}{f}\frac{d^2f}{dx^2} = -k_1^2, \qquad \frac{1}{g}\frac{d^2g}{dy^2} = -k_2^2, \qquad \frac{1}{h}\frac{d^2h}{dz^2} = k_1^2 + k_2^2 = k_3^2$$

I made the choice of the signs for these constants because the boundary function is periodic in x and in y, so I expect sines and cosines along those directions. The separated solution is

$$(A\sin k_1 x + B\cos k_1 x)(C\sin k_2 y + D\cos k_2 y)(Ee^{k_3 z} + Fe^{-k_3 z}) \tag{10.47}$$

What about the case for separation constants of zero? Yes, that's needed too; the average value of the potential on the surface is $V_0/2$, so just as with the example leading to Eq. (10.43) this will have a constant term of that value. The periodicity in x is $2a$ and in y it is $2b$, so this determines

$$k_1 = n\pi/a, \qquad k_2 = m\pi/b \qquad \text{then} \qquad k_3 = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}, \quad n,\, m = 1,\, 2, \ldots$$

where n and m are *independent* integers. Use the experience that led to Eq. (10.45) to write V on the surface as a sum of the constant $V_0/2$ and a function that is odd in both x and in y. As there, the odd function in x will be represented by a sum of sines in x, and the same statement will hold for the y coordinate. This leads to the form of the sum

$$V(x, y, z) = \frac{1}{2}V_0 + \sum_{n=1}^\infty \sum_{m=1}^\infty \alpha_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-k_{nm} z}$$

where k_{nm} is the k_3 of the preceding equation. What happened to the other term in z, the one with the positive exponent? Did I say that I'm looking for solutions in the domain $z > 0$?

At $z = 0$ this must match the boundary conditions stated, and as before, the orthogonality of the sines on the two domains allows you to determine the coefficients. You simply have to do two integrals instead of one. See problem 10.19.

$$V(x, y, z > 0) = \frac{1}{2}V_0 + \frac{8V_0}{\pi^2}\sum_{\text{odd } n}^{\infty}\sum_{\text{odd } m}^{\infty}\frac{1}{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-k_{nm}z} \tag{10.48}$$

10.7 Cylindrical Coordinates

Rectangular coordinates are not always the right choice. Cylindrical, spherical, and other choices are often needed. For cylindrical coordinates, the gradient and divergence are, from Eqs. (9.24) and (9.15)

$$\nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{z}\frac{\partial V}{\partial z} \qquad \text{and} \qquad \nabla\cdot\vec{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Combine these to get the Laplacian in cylindrical coordinates.

$$\nabla\cdot\nabla V = \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \tag{10.49}$$

For electrostatics, the equation remains $\nabla^2 V = 0$, and you can approach it the same way as before, using separation of variables. I'll start with the special case for which everything is independent of z. Assume a solution of the form $V = f(r)g(\phi)$, then

$$\nabla^2 V = g\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + f\frac{1}{r^2}\frac{\partial^2 g}{\partial \phi^2} = 0$$

Multiply this by r^2 and divide by $f(r)g(\phi)$ to get

$$\frac{r}{f}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{g}\frac{\partial^2 g}{\partial \phi^2} = 0$$

This is separated. The first terms depends on r alone, and the second term on ϕ alone. For this to hold the terms must be constants.

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) = \alpha \qquad \text{and} \qquad \frac{1}{g}\frac{d^2 g}{d\phi^2} = -\alpha \tag{10.50}$$

The second equation, in ϕ, is familiar. If α is positive this is a harmonic oscillator, and that is the most common way this solution is applied. I'll then look at the case for $\alpha \geq 0$, for which the substitution $\alpha = n^2$ makes sense.

$$\alpha = 0: \qquad \frac{d^2 g}{d\phi^2} = 0 \implies g(\phi) = A + B\phi$$

$$r\frac{d}{dr}\left(r\frac{df}{dr}\right) = 0 \implies f(r) = C + D\ln r$$

$$\alpha = n^2 > 0 \qquad \frac{d^2g}{d\phi^2} = -n^2 g \implies g(\phi) = A\cos n\phi + B\sin n\phi$$
$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} - n^2 f = 0 \implies f(r) = Cr^n + Dr^{-n}$$

There's not yet a restriction that n is an integer, though that's often the case. Verifying the last solution for f is easy.

A general solution that is the sum of all these terms is

$$V(r,\phi) = (C_0 + D_0\ln r)(A_0 + B_0\phi) + \sum_n (C_n r^n + D_n r^{-n})(A_n\cos n\phi + B_n\sin n\phi) \qquad (10.51)$$

Some of these terms should be familiar:
C_0A_0 is just a constant potential.
$D_0A_0\ln r$ is the potential of a uniform line charge; $d\ln r/dr = 1/r$, and that is the way that the electric field drops off with distance from the axis.
$C_1A_1 r\cos\phi$ is the potential of a uniform field (as is the $r\sin\phi$ term). Write this in the form $C_1A_1 r\cos\phi = C_1A_1 x$, and the gradient of this is $C_1A_1\hat{x}$. The sine gives $\hat{y}$.
See problem 10.24.

Example

A conducting wire, radius R, is placed in a uniform electric field $\vec{E}_0$, and perpendicular to it. Put the wire along the z-axis and call the positive x-axis the direction that the field points. That's $\phi = 0$. In the absence of the wire, the potential for the uniform field is $V = -E_0 x = -E_0 r\cos\phi$, because $-\nabla V = E_0\hat{x}$. The total solution will be in the form of Eq. (10.51).

Now turn the general form of the solution into a particular one for this problem. The entire range of ϕ from 0 to 2π appears here; you can go all the way around the origin and come back to where you started. The potential is a function, meaning that it's single valued, and that eliminates $B_0\phi$. It also implies that all the n are integers. The applied field has a potential that is even in ϕ. That means that you should expect the solution to be even in ϕ too. Does it really? You also have to note that the physical system and its attendant boundary conditions are even in ϕ — it's a cylinder. Then too, the first few times that you do this sort of problem you should see what happens to the odd terms; what makes them vanish? I won't eliminate the $\sin\phi$ terms from the start, but I'll calculate them and show that they do vanish.

$$V(r,\phi) = (C_0 + D_0\ln r)B_0 + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})(A_n\cos n\phi + B_n\sin n\phi)$$

Carrying along all these products of (still unknown) factors such as D_nA_n is awkward. It makes it look neater and it is easier to follow if I combine and rename some of them.

$$V(r,\phi) = C_0 + D_0\ln r + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})\cos n\phi + \sum_{n=1}^{\infty}(C'_n r^n + D'_n r^{-n})\sin n\phi \qquad (10.52)$$

As $r\to\infty$, the potential looks like $-E_0 r\cos\phi$. That implies that $C_n = 0$ for $n > 1$, and that $C'_n = 0$ for all n, and that $C_1 = -E_0$.

Now use the boundary conditions on the cylinder. It is a conductor, so in this static case the potential is a constant all through it, in particular on the surface. I may as well take that constant to be zero, though it doesn't really matter.

$$V(R,\phi) = 0 = C_0 + D_0\ln R + \sum_{n=1}^{\infty}(C_n R^n + D_n R^{-n})\cos n\phi + \sum_{n=1}^{\infty}(C'_n R^n + D'_n R^{-n})\sin n\phi$$

Multiply by $\sin m\phi$ and integrate over ϕ. The trigonometric functions are orthogonal, so all that survives is

$$0 = (C'_m R^m + D'_m R^{-m})\pi \qquad \text{all } m \geq 1$$

That gets rid of all the rest of the sine terms as promised: $D'_m = 0$ for all m because $C'_m = 0$ for all m. Now repeat for $\cos m\phi$.

$$0 = C_0 + D_0 \ln R \quad (m = 0) \qquad \text{and} \qquad 0 = (C_m R^m + D_m R^{-m})\pi \quad (m > 0)$$

All of the $C_m = 0$ for $m > 1$, so this says that the same applies to D_m. The $m = 1$ equation determines D_1 in terms of C_1 and then E_0.

$$D_1 = -C_1 R^2 = +E_0 R^2$$

Only the C_0 and D_0 terms are left, and that requires another boundary condition. When specifying the problem initially, I didn't say whether or not there is any charge on the wire. In such a case you could naturally assume that it is zero, but you have to say so explicitly because that affects the final result. Make it zero. That kills the $\ln r$ term. The reason for that goes back to the interpretation of this term. Its negative gradient is the electric field, and that would be $-1/r$, the field of a uniform line charge. If I assume there isn't one, then $D_0 = 0$ and so the same for C_0. Put this all together and

$$V(r, \phi) = -E_0 r \cos\phi + E_0 \frac{R^2}{r} \cos\phi \tag{10.53}$$

The electric field that this presents is, from Eq. (9.24)

$$\begin{aligned}\vec{E} = -\nabla V &= E_0\big(\hat{r}\cos\phi - \hat{\phi}\sin\phi\big) - E_0 R^2 \left(-\hat{r}\frac{1}{r^2}\cos\phi - \hat{\phi}\frac{1}{r^2}\sin\phi\right) \\ &= E_0\,\hat{x} + E_0\frac{R^2}{r^2}\big(\hat{r}\cos\phi + \hat{\phi}\sin\phi\big)\end{aligned}$$

As a check to see what this looks like, what is the electric field at the surface of the cylinder?

$$\vec{E}(R, \phi) = E_0\big(\hat{r}\cos\phi - \hat{\phi}\sin\phi\big) - E_0 R^2 \left(-\hat{r}\frac{1}{R^2}\cos\phi - \hat{\phi}\frac{1}{R^2}\sin\phi\right) = 2E_0\,\hat{r}\cos\phi$$

It's perpendicular to the surface, as it should be. At the left and right, $\phi = 0, \pi$, it is twice as large as the uniform field alone would be at those points.

Problems

10.1 The specific heat of a particular type of stainless steel (CF8M) is 500 J/kg·K. Its thermal conductivity is 16.2 W/m·K and its density is $7750\,\mathrm{kg/m^3}$. A slab of this steel 1.00 cm thick is at a temperature $100°$C and it is placed into ice water. Assume the simplest boundary condition that its surface temperature stays at zero, and find the internal temperature at later times. When is the 2^{nd} term in the series, Eq. (10.15), only 5% of the 1^{st}? Sketch the temperature distribution then, indicating the scale correctly.

10.2 In Eq. (10.13) I eliminated the $n = 0$ solution by a fallacious argument. What is α in this case? This gives one more term in the sum, Eq. (10.14). Show that with the boundary conditions stated, this extra term is zero anyway (this time).

10.3 In Eq. (10.14) you have the sum of many terms. Does it still satisfy the original differential equation, Eq. (10.3)?

10.4 In the example Eq. (10.15) the final temperature was zero. What if the final temperature is T_1? Or what if I use the Kelvin scale, so that the final temperature is $273°$? Add the appropriate extra term, making sure that you still have a solution to the original differential equation and that the boundary conditions are satisfied.

10.5 In the example Eq. (10.15) the final temperature was zero on both sides. What if it's zero on just the side at $x = 0$ while the side at $x = L$ stays at T_0? What is the solution now?
Ans: $T_0 x/L + (2T_0/\pi)\sum_1^\infty (1/n)\sin(n\pi x/L)e^{-n^2\pi^2 Dt/L^2}$

10.6 You have a slab of material of thickness L and at a uniform temperature T_0. The side at $x = L$ is insulated so that heat can't flow in or out of that surface. By Eq. (10.1), this tells you that $\partial T/\partial x = 0$ at that surface. Plunge the other side into ice water at temperature $T = 0$ and find the temperature inside at later time. The boundary condition on the $x = 0$ surface is the same as in the example in the text, $T(0, t) = 0$. (**a**) Separate variables and find the appropriate separated solutions for these boundary conditions. Are the separated solutions orthogonal? Use the techniques of Eq. (5.15). (**b**) When the lowest order term has dropped to where its contribution to the temperature at $x = L$ is $T_0/2$, how big is the next term in the series? Sketch the temperature distribution in the slab at that time.
Ans: $(4T_0/\pi)\sum_0^\infty (\frac{1}{2n+1})\sin\left[(n+\frac{1}{2})\pi x/L\right]e^{-(n+1/2)^2\pi^2 Dt/L^2}$, $-9.43\times 10^{-5}T_0$

10.7 In the analysis leading to Eq. (10.26) the temperature at $y = b$ was set to T_0. If instead, you have the temperature at $x = a$ set to T_0 with all the other sides at zero, write down the answer for the temperature within the rod. Now use the fact that Eq. (10.21) is linear to write down the solution if *both* the sides at $y = b$ and $x = a$ are set to T_0.

10.8 In leading up to Eq. (10.25) I didn't examine the third possibility for the separation constant, that it's zero. Do so.

10.9 Look at the boundary condition of Eq. (10.22) again. Another way to solve this problem is to use the solution for which the separation constant is zero, and to use it to satisfy the conditions at $y = 0$ and $y = b$. You will then have one term in the separated solution that is $T_0 y/b$, and that means that in Eq. (10.23) you will have to choose the separation variable to be positive instead of negative. Why? Because now all the rest of the terms in the sum over separated solutions must vanish at $y = 0$ and $y = b$. You've already satisfied the boundary conditions on those surfaces by using the $T_0 y/b$ term. Now you have to satisfy the boundary conditions on $x = 0$ and $x = a$ because the total temperature

there must be zero. That in turn means that the sum over all the rest of the separated terms must add to $-T_0 y/b$ at $x = 0$ and $x = a$. When you analyze this solution in the same spirit as the analysis of Eq. (10.26), compare the convergence properties of that solution to your new one. In particular, look at $a \ll b$ and $a \gg b$ to see which version converges better in each case.
Ans: $T_0 y/b + (2T_0/\pi)\sum_1^\infty \big[(-1)^n/n\big]\sin(n\pi y/b)\cosh\big(n\pi(x-a/2)/b\big)/\cosh(n\pi a/2b)$

10.10 Finish the re-analysis of the electrostatic boundary value problem Eq. (10.45) starting from Eq. (10.46). This will get the potential for $z \neq 0$ with perhaps less work than before.

10.11 Examine the solution Eq. (10.42) at $z = 0$ in the light of problem 5.11.

10.12 A thick slab of material is alternately heated and cooled at its surface so the its surface temperature oscillates as

$$T(0,t) = \begin{cases} T_1 & (0 < t < t_0) \\ -T_1 & (t_0 < t < 2t_0) \end{cases} \qquad T(0, t + 2t_0) = T(0,t)$$

That is, the period of the oscillation is $2t_0$. Find the temperature inside the material, for $x > 0$. How does this behavior differ from the solution in Eq. (10.20)?
Ans: $\frac{4T_1}{\pi}\sum_{k=0}^\infty \big(1/(2k+1)\big)e^{-\beta_k x}\sin\big((2k+1)\omega t - \beta_k x\big);\ \ \omega = \pi/t_0$

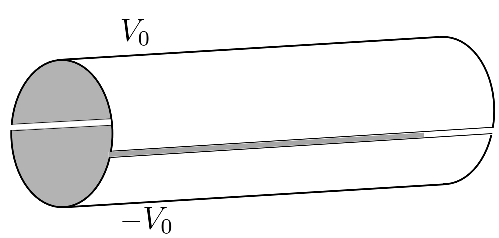
10.13 Fill in the missing steps in finding the solution, Eq. (10.34).

10.14 A variation on the problem of the alternating potential strips in section 10.6. Place a grounded conducting sheet parallel to the x-y plane at a height $z = d$ above it. The potential there is then $V(x,y,z=d) = 0$. Solve for the potential in the gap between $z = 0$ and $z = d$. A suggestion: you may find it easier to turn the coordinate system over so that the grounded sheet is at $z = 0$ and the alternating strips are at $z = d$. This switch of coordinates is in no way essential, but it is a bit easier. Also, I want to point out that you will need to consider the case for which the separation constant in Eq. (10.37) is zero.

10.15 Starting from Eq. (10.52) and repeat the example there, but assume that the conducting wire is in an external electric field $E_0\hat{y}$ instead of $E_0\hat{x}$. Repeat the calculation for the potential and for the electric field, filling in the details of the missing steps.

10.16 A very long conducting cylindrical shell of radius R is split in two along lines parallel to its axis. The two halves are wired to a circuit that places one half at potential V_0 and the other half at potential $-V_0$. (**a**) What is the potential everywhere inside the cylinder? Use the results of section 10.7 and assume a solution of the form

$$V(r,\phi) = \sum_0^\infty r^n\big(a_n\cos n\phi + b_n\sin n\phi\big)$$



Match the boundary condition that

$$V(R,\phi) = \begin{cases} V_0 & (0 < \phi < \pi) \\ -V_0 & (\pi < \phi < 2\pi) \end{cases}$$

I picked the axis for $\phi = 0$ pointing toward the split between the cylinders. No particular reason, but you have to make a choice. I make the approximation that the cylinder is infinitely long so that z dependence doesn't enter. Also, the two halves of the cylinder almost touch so I'm neglecting the distance between them.
(**b**) What is the electric field, $-\nabla V$ on the central axis? Is this answer more or less what you would estimate before solving the problem? Ans: (**b**) $E = 4V_0/\pi R$.

10.17 Solve the preceding problem *outside* the cylinder. The integer n can be either positive or negative, and this time you'll need the negative values. (And *why* must n be an integer?)
Ans: $(4V_0/\pi)\sum_{n\text{ odd}}(1/n)(R/r)^n \sin n\phi$

10.18 In the split cylinder of problem 10.16, insert a coaxial wire of radius $R_1 < R$. It is at zero potential. Now what is the potential in the domain $R_1 < r < R$? You will need *both* the positive and negative n values, $\sum(A_n r^n + B_n r^{-n})\sin n\phi$
Ans: $(4V_0/\pi)\sum_{m\text{ odd}}\sin m\phi[-R_1^{-m}r^m + R_1^m r^{-m}]/m[R^{-m}R_1^m - R_1^{-m}R^m]$

10.19 Fill in the missing steps in deriving Eq. (10.48).

10.20 Analyze how rapidly the solution Eq. (10.48) approaches a constant as z increases from zero. Compare Eq. (10.44).

10.21 A broad class of second order linear homogeneous differential equations can, with some manipulation, be put into the form (Sturm-Liouville)

$$(p(x)u')' + q(x)u = \lambda w(x)u$$

Assume that the functions p, q, and w are real, and use manipulations much like those that led to the identity Eq. (5.15). Derive the analogous identity for this new differential equation. When you use separation of variables on equations involving the Laplacian you will commonly come to an ordinary differential equation of exactly this form. The precise details will depend on the coordinate system you are using as well as other aspects of the PDE.

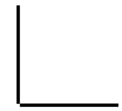
10.22 Carry on from Eq. (10.31) and deduce the separated solution that satisfies these boundary condition. Show that it is equivalent to Eq. (10.32).

10.23 The Laplacian in cylindrical coordinates is Eq. (10.49). Separate variables for the equation $\nabla^2 V = 0$ and you will see that the equations in z and ϕ are familiar. The equation in the r variable is less so, but you've seen it (almost) in Eqs. (4.18) and (4.22). Make a change of variables in the r-differential equation, $r = kr'$, and turn it into exactly the form described there.

10.24 In the preceding problem suppose that there's no z-dependence. Look at the case where the separation constant is zero for both the r and ϕ functions, finally assembling the product of the two for another solution of the whole equation.
These results provide four different solutions, a constant, a function of r alone, a function of ϕ alone, and a function of both. In each of these cases, assume that these functions are potentials V and that $\vec{E} = -\nabla V$ is the electric field from each potential. Sketch equipotentials for each case, then sketch the corresponding vector fields that they produce (a lot of arrows).

10.25 Do problem 8.23 and now solve it, finding all solutions to the wave equation.
Ans: $f(x - vt) + g(x + vt)$

10.26 Use the results of problem 10.24 to find the potential in the corner between two very large metal plates set at right angles. One at potential zero, the other at potential V_0. Compute the electric field, $-\nabla V$ and draw the results. Ans: $-2V_0\,\hat{\phi}/\pi r$



10.27 A thin metal sheet has a straight edge for one of its boundaries. Another thin metal sheet is cut the same way. The two straight boundaries are placed in the same plane and almost, but not quite touching. Now apply a potential difference between them, putting one at a voltage V_0 and the other at $-V_0$. *In the region of space near to the almost touching boundary,* what is the electric potential? From that, compute and draw the electric field.

10.28 A slab of heat conducting material lies between coordinates $x = -L$ and $x = +L$, which are at temperatures T_1 and T_2 respectively. In the steady state ($\partial T/\partial t \equiv 0$), what is the temperature distribution inside? Now express the result in cylindrical coordinates around the z-axis and show how it matches the sum of cylindrical coordinate solutions of $\nabla^2 T = 0$ from problem 10.15. What if the surfaces of the slab had been specified at $y = -L$ and $y = +L$ instead?

10.29 The result of problem 10.16 has a series of terms that look like $(x^n/n)\sin n\phi$ (odd n). You can use complex exponentials, do a little rearranging and factoring, and sum this series. Along the way you will have to figure out what the sum $z + z^3/3 + z^5/5 + \cdots$ is. Refer to section 2.7. Finally of course, the answer is real, and if you look hard you may find a simple interpretation for the result. Be sure you've done problem 10.24 before trying this last step. Ans: $2V_0(\theta_1 + \theta_2)/\pi$. You still have to decipher what θ_1 and θ_2 are.

10.30 Sum the series Eq. (10.27) to get a closed-form analytic expression for the temperature distribution. You will find the techniques of section 5.7 useful, but there are still a lot of steps. Recall also $\ln(r\,e^{i\theta}) = \ln r + i\theta$. Ans: $(2T_0/\pi)\tan^{-1}\big[\sin(\pi x/a)/\sinh\big(\pi(b-y)/a\big)\big]$

10.31 A generalization of the problem specified in Eq. (10.22). Now the four sides have temperatures given respectively to be the constants T_1, T_2, T_3, T_4. Note: with a little bit of foresight, you won't have to work very hard at all to solve this.

10.32 Use the electrostatic equations from problem 9.21 and assume that the electric charge density is given by $\rho = \rho_0 a/r$, where this is in cylindrical coordinates. (**a**) What cylindrically symmetric electric field comes from this charge distribution? (**b**) From $\vec{E} = -\nabla V$ what potential function V do you get?

10.33 Repeat the preceding problem, but now interpret r as referring to spherical coordinates. What is $\nabla^2 V$?

10.34 The Laplacian in spherical coordinates is Eq. (9.43). The electrostatic potential equation is $\nabla^2 V = 0$ just as before, but now take the special case of azimuthal symmetry so that the potential function is independent of ϕ. Apply the method of separation of variables to find solutions of the form $f(r)g(\theta)$. You will get two ordinary differential equations for f and g. The second of these equations is much simpler if you make the change of independent variable $x = \cos\theta$. Use the chain rule a couple of times to do so, showing that the two differential equations are

$$(1 - x^2)\frac{d^2 g}{dx^2} - 2x\frac{dg}{dx} + Cg = 0 \qquad \text{and} \qquad r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - Cf = 0$$

10.35 For the preceding equations show that there are solutions of the form $f(r) = Ar^n$, and recall the analysis in section 4.11 for the g solutions. What values of the separation constant C will allow solutions that are finite as $x \to \pm 1$ $(\theta \to 0,\, \pi)$? What are the corresponding functions of r? Don't forget that there are two solutions to the second order differential equation for f — two roots to a quadratic equation.

10.36 Write out the separated solutions to the preceding problem (the ones that are are finite as θ approaches 0 or π) for the two smallest allowed values of the separation constant C: 0 and 2. In each of the four cases, interpret and sketch the potential and its corresponding electric field, $-\nabla V$. How do you sketch a potential? Draw equipotentials.

10.37 From the preceding problem you can have a potential, a solution of Laplace's equation, in the form $\big(Ar + B/r^2\big)\cos\theta$. Show that by an appropriate choice of A and B, this has an electric field that for large distances from the origin looks like $E_0\,\hat z$, and that on the sphere $r = R$ the total potential is zero — a grounded, conducting sphere. What does the total electric field look like for $r > R$; sketch some field lines. Start by asking what the electric field is as $r \to R$.

10.38 Similar to problem 10.16, but the potential on the cylinder is

$$V(R,\phi) = \begin{cases} V_0 & (0 < \phi < \pi/2 \text{ and } \pi < \phi < 3\pi/2) \\ -V_0 & (\pi/2 < \phi < \pi \text{ and } 3\pi/2 < \phi < 2\pi) \end{cases}$$

Draw the electric field in the region near $r = 0$.

10.39 What is the area charge density on the surface of the cylinder where the potential is given by Eq. (10.53)?